

Closing *Thurs*: HW_6A,6B (7.4, 7.5)
Closing next *Wed*: HW_7A,7B (7.5, 7.7, 7.8)
Note: Friday is a university holiday (no class)
Exam 2 is next **Thursday** (Nov. 16th)
Covers 6.4, 6.5, 7.1-7.5, 7.7, 7.8

We have learned how to integral some important situations. But despite our best efforts in 7.1-7.5, many, many, many integrals CANNOT be done with any of our methods. So, in many applications, we have to approximate!

7.7 Approximating Integrals

To approximate $\int_a^b f(x)dx$

1. Compute $\Delta x = \frac{b-a}{n}$.

Label the tick marks: $x_i = a + i\Delta x$

2. Use an approximation method:

$$L_n = \Delta x [f(x_0) + f(x_1) + \cdots + f(x_{n-1})] \quad (\text{Left endpoint})$$

$$R_n = \Delta x [f(x_1) + f(x_2) + \cdots + f(x_n)] \quad (\text{Right endpoint})$$

$$M_n = \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)] \quad (\text{Midpoint})$$

New - Trapezoid Rule: (all the “middle terms” are multiplied by 2)

$$T_n = \frac{1}{2} \Delta x [f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

New - Simpson's Rule: n must be even! (Alternating multiplying middle terms by 4 and 2)

$$S_n = \frac{1}{3} \Delta x [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Example: (None of our methods can integrate this)

Estimate $\int_0^3 \sqrt{100 - x^3} dx$, using $n = 3$ subdivisions.

- Step 1: $\Delta x = \frac{3-0}{3} = 1$. $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

Step 2: Here is each method:

$$\int_0^3 \sqrt{100 - x^3} dx \approx L_3 = (1) \left[\sqrt{100 - (0)^3} + \sqrt{100 - (1)^3} + \sqrt{100 - (2)^3} \right] \approx 29.5415$$

$$\int_0^3 \sqrt{100 - x^3} dx \approx R_3 = (1) \left[\sqrt{100 - (1)^3} + \sqrt{100 - (2)^3} + \sqrt{100 - (3)^3} \right] \approx 28.0855$$

$$\int_0^3 \sqrt{100 - x^3} dx \approx M_3 = (1) \left[\sqrt{100 - (0.5)^3} + \sqrt{100 - (1.5)^3} + \sqrt{100 - (2.5)^3} \right] \approx 29.0091$$

VEW – Trapezoid rule with $n = 3$.

$$T_3 = \frac{1}{2} (1) \left[\sqrt{100 - (0)^3} + 2\sqrt{100 - (1)^3} + 2\sqrt{100 - (2)^3} + \sqrt{100 - (3)^3} \right] \approx 28.8135$$

VEW – Simpson's rule with $n = 6$ (n must be even), $\Delta x = \frac{3-0}{6} = \frac{1}{2}$. $x_0 = 0, x_1 = \frac{1}{2}, \dots$

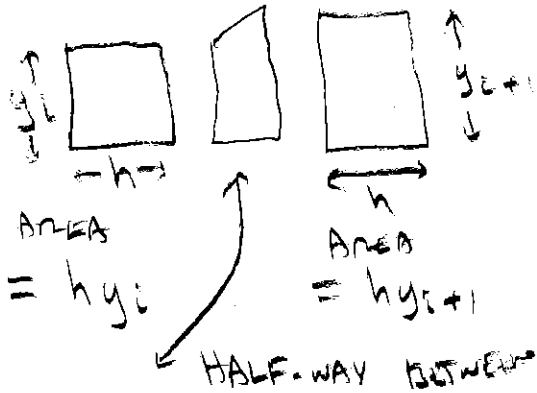
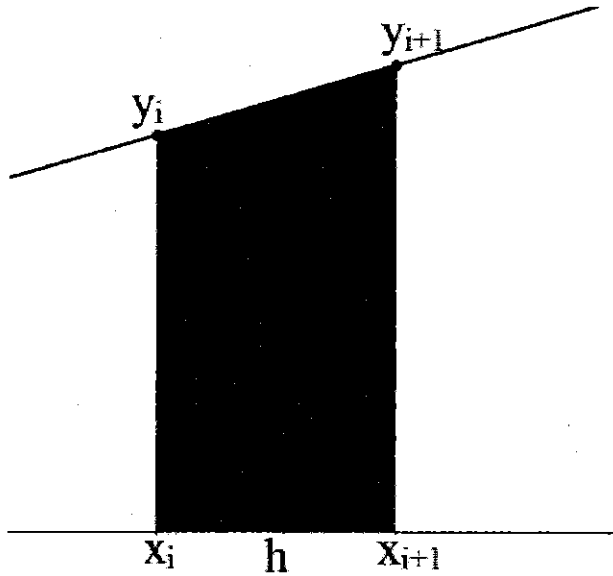
$$S_6 = \frac{1}{3} \cdot \frac{1}{2} \left[\sqrt{100 - (0)^3} + 4\sqrt{100 - (0.5)^3} + 2\sqrt{100 - (1)^3} + 4\sqrt{100 - (1.5)^3} \right. \\ \left. + 2\sqrt{100 - (2)^3} + 4\sqrt{100 - (2.5)^3} + \sqrt{100 - (3)^3} \right] \approx 28.9441$$

Actual Value (to 8 places after the decimal): 28.94418784

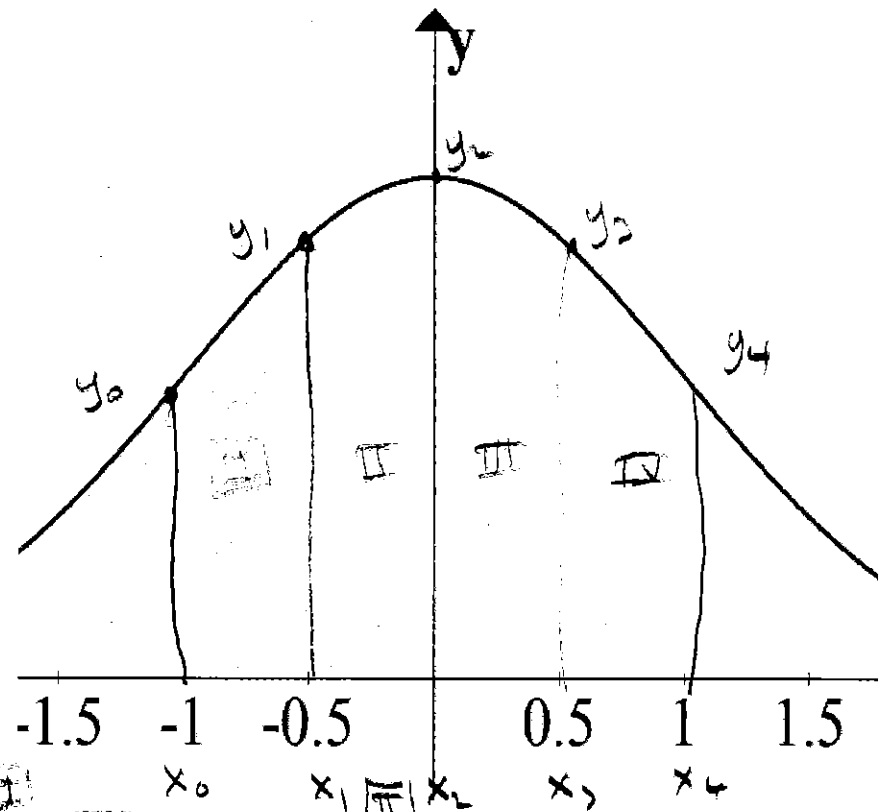
7.7 Derivation Notes

Trapezoid Rule:

$$\text{Shaded Area} = \frac{h}{2} (y_i + y_{i+1})$$



$$\frac{1}{2} (h y_i + h y_{i+1}) = \frac{1}{2} h (y_i + y_{i+1})$$



$$\frac{1}{2} \Delta x (y_0 + y_1) + \frac{1}{2} \Delta x (y_1 + y_2) + \frac{1}{2} \Delta x (y_2 + y_3) + \frac{1}{2} \Delta x (y_3 + y_4)$$

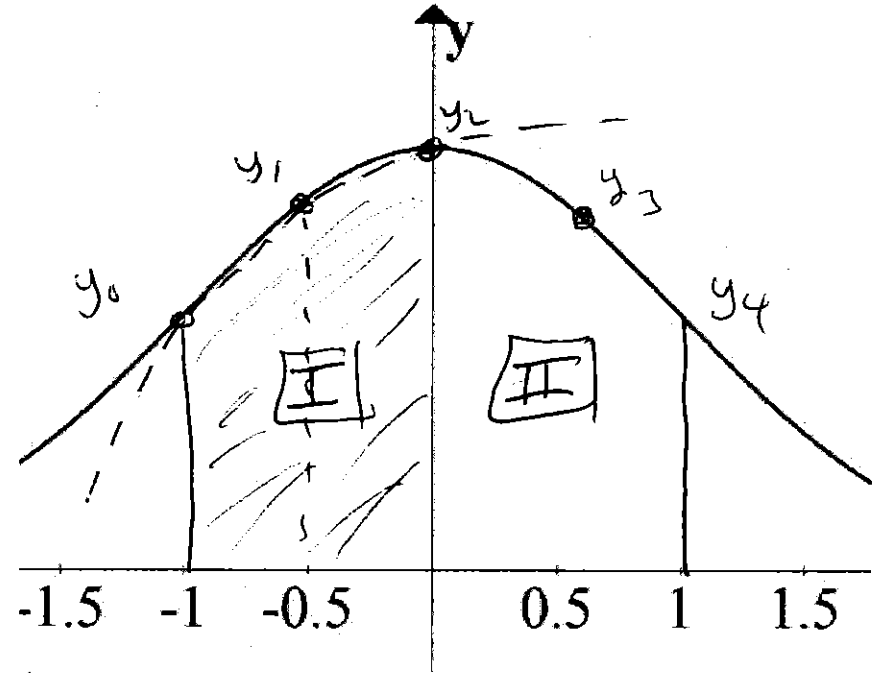
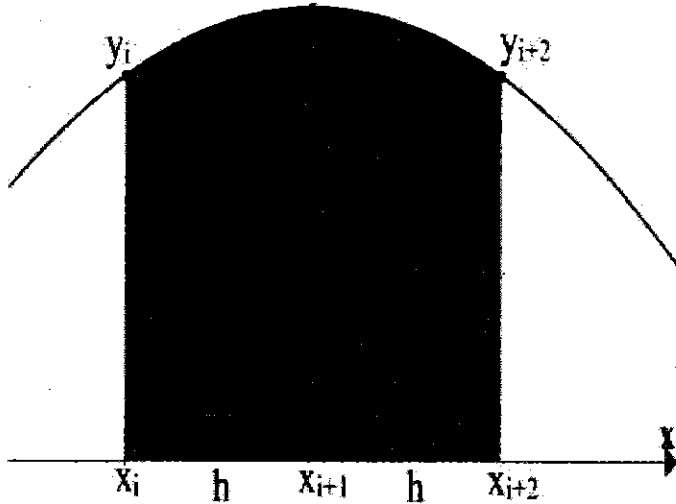
$$\frac{1}{2} \Delta x [y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

Simpson's Rule:

If the curve below is a **parabola**,

$y = ax^2 + bx + c$, that goes through the three indicated points, then

$$\text{Shaded Area} = \frac{h}{3} (y_i + 4y_{i+1} + y_{i+2})$$



$$\frac{1}{3} \Delta x (y_0 + 4y_1 + y_2) + \frac{1}{3} \Delta x (y_2 + 4y_3 + y_4)$$

$$\int_{x_i}^{x_{i+2}} ax^2 + bx + c dx = \left. \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx \right|_{x_i}^{x_{i+2}}$$

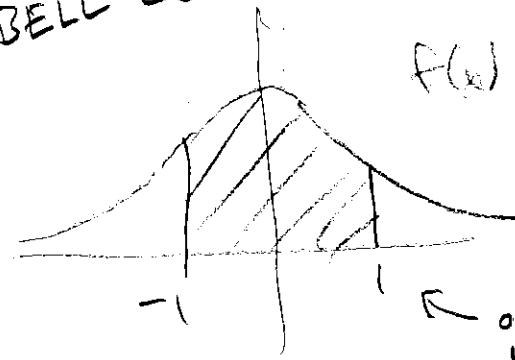
$$\frac{1}{3} \Delta x (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

Example:

With $n = 4$, use both new methods to approximate (just set up)

$$\frac{1}{\sqrt{2\pi}} \int_{-1}^1 e^{-\frac{1}{2}x^2} dx$$

BELL CURVE!



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

$$\Delta x = \frac{1-(-1)}{4} = \frac{1}{2}, \quad x_0 = -1, \quad x_1 = -\frac{1}{2}, \quad x_2 = 0, \quad x_3 = \frac{1}{2}, \quad x_4 = 1$$

$$\frac{1}{2} \Delta x [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2} \left(\frac{1}{2}\right) \left[e^{-\frac{1}{2}(-1)^2} + 2e^{-\frac{1}{2}(-\frac{1}{2})^2} + 2e^{-\frac{1}{2}(0)^2} + 2e^{-\frac{1}{2}(\frac{1}{2})^2} + e^{-\frac{1}{2}(1)^2} \right] \approx 0.672518$$

$$\frac{1}{3} \Delta x [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{3} \left(\frac{1}{2}\right) \left[e^{-\frac{1}{2}(-1)^2} + 4e^{-\frac{1}{2}(-\frac{1}{2})^2} + 2e^{-\frac{1}{2}(0)^2} + 4e^{-\frac{1}{2}(\frac{1}{2})^2} + e^{-\frac{1}{2}(1)^2} \right] \approx 0.6827109757$$

"ACTUAL" VALUE = 0.6826894921

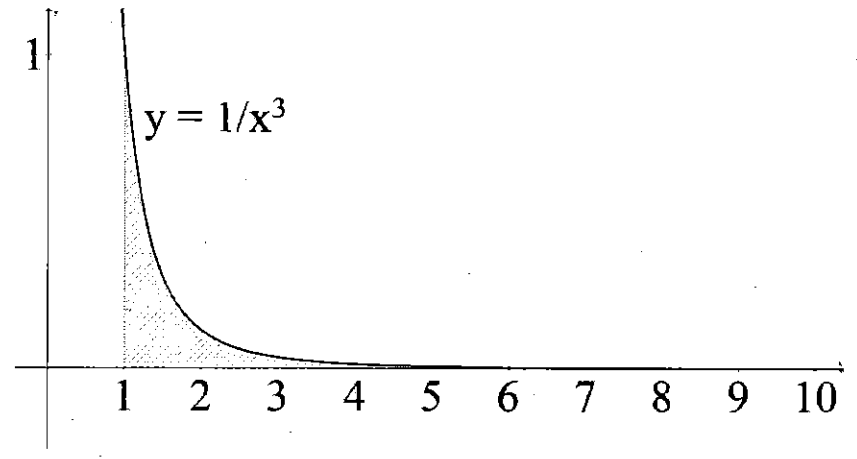
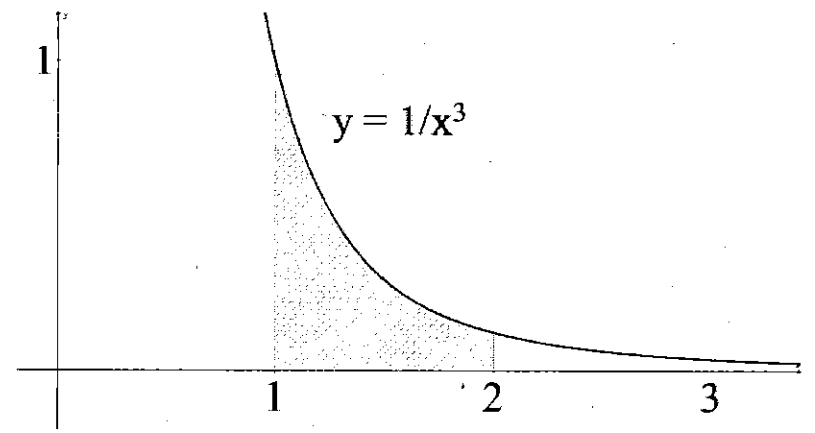
7.8 Improper Integrals (Preview)

Motivation:

Consider the function $f(x) = \frac{1}{x^3}$.

Compute the area under the function...

1. ...from $x = 1$ to $x = t$
2. ...from $x = 1$ to $x = 10$
3. ...from $x = 1$ to $x = 100$



$$\begin{aligned} \int_1^t \frac{1}{x^3} dx &= \int_1^t x^{-3} dx \\ &= \left. -\frac{1}{2} x^{-2} \right|_1^t \\ &= -\frac{1}{2} \frac{1}{t^2} - \left(-\frac{1}{2} \right) \\ &= \boxed{-\frac{1}{2} \frac{1}{t^2} + \frac{1}{2}} \end{aligned}$$

$$\text{So } \int_1^{10} \frac{1}{x^3} dx = -\frac{1}{2} \frac{1}{10^2} + \frac{1}{2} = 0.495$$

$$\int_1^{100} \frac{1}{x^3} dx = -\frac{1}{2} \frac{1}{100^2} + \frac{1}{2} = 0.49995$$

Def'n: Improper type 1 -

infinite integral of integration

$$\int_a^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$$

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$$

If the limit exists and is finite, then we say the integral *converges*.

Otherwise, we say it *diverges*.

Example:

$$\int_0^{\infty} \frac{1}{x^3} dx = \lim_{t \rightarrow \infty} \left[\int_1^t \frac{1}{x^3} dx \right]$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{2} \frac{1}{t^2} + \frac{1}{2} \right]$$

$$= 0 + \frac{1}{2} = \boxed{\frac{1}{2}}$$

CONVERGES!

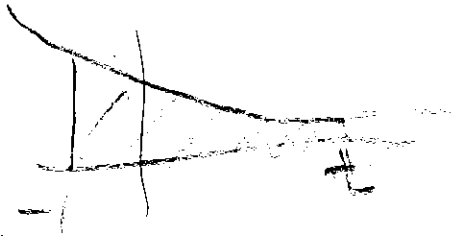
Example:

$$\begin{aligned}\int_{-1}^{\infty} e^{-2x} dx &= \lim_{t \rightarrow \infty} \int_{-1}^t e^{-2x} dx \\ &= \lim_{t \rightarrow \infty} \left[\frac{1}{-2} e^{-2x} \Big|_{-1}^t \right] \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{2} e^{-2t} - -\frac{1}{2} e^{2} \right] \\ &= \lim_{t \rightarrow \infty} \left[-\frac{1}{2} e^{-2t} + \frac{1}{2} e^2 \right]\end{aligned}$$

Since as $t \rightarrow \infty$, $e^{-2t} \rightarrow 0$

$$= 0 + \frac{1}{2} e^2 = \frac{1}{2} e^2$$

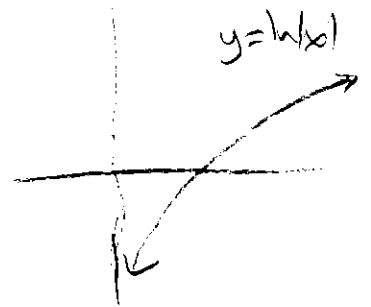
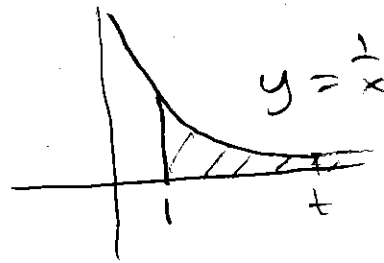
CONVERGES



Example:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{t \rightarrow \infty} \left[\int_1^t \frac{1}{x} dx \right] \\ &= \lim_{t \rightarrow \infty} \left[\ln|x| \Big|_1^t \right] \\ &= \lim_{t \rightarrow \infty} \left[\ln(t) - \underbrace{\ln(1)}_0 \right] \\ &= \lim_{t \rightarrow \infty} \ln(t) = \infty\end{aligned}$$

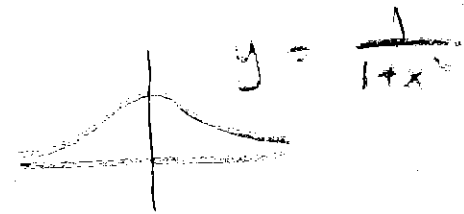
DIVERGES



Def'n:

$$\int_{-\infty}^{\infty} f(x) dx = \lim_{r \rightarrow -\infty} \int_r^0 f(x) dx + \lim_{t \rightarrow \infty} \int_0^t f(x) dx$$

In this case, we say it *converges* only if both limits separately exist and are finite.



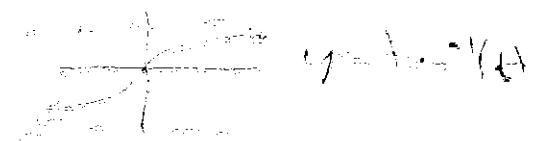
Example:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$$

$$= \lim_{r \rightarrow -\infty} \left[\int_r^0 \frac{1}{1+x^2} dx \right] + \lim_{t \rightarrow \infty} \left[\int_0^t \frac{1}{1+x^2} dx \right]$$

$$= \lim_{r \rightarrow -\infty} \left[\underbrace{\tan^{-1}(0)}_0 - \tan^{-1}(r) \right] + \lim_{t \rightarrow \infty} \left[\tan^{-1}(t) - \underbrace{\tan^{-1}(0)}_0 \right]$$

$$= -(-\pi/2) + \pi/2 = \boxed{\pi}$$



**Def'n: Improper type 2 -
infinite discontinuity**

If $f(x)$ has a discontinuity at $x = a$, then

$$\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$$

If $f(x)$ has a discontinuity at $x = b$, then

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

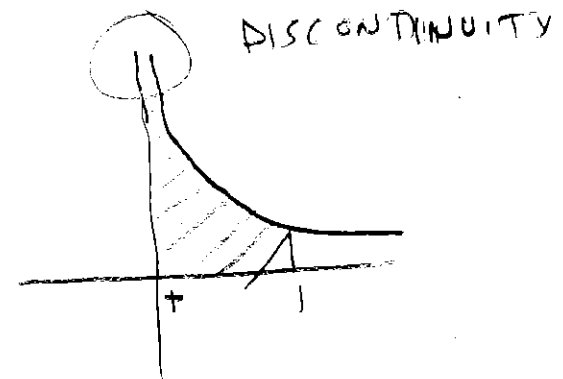
If the limit exists and is finite, then we say the integral *converges*.

Otherwise, we say it *diverges*.

Example:

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt{x}} dx &= \lim_{t \rightarrow 0^+} \left[\int_t^1 x^{-1/2} dx \right] \\ &= \lim_{t \rightarrow 0^+} \left[2x^{1/2} \Big|_t^1 \right] \\ &= \lim_{t \rightarrow 0^+} \left[2\sqrt{1} - 2\sqrt{t} \right] \\ &= 2 - 0 = \boxed{2} \end{aligned}$$

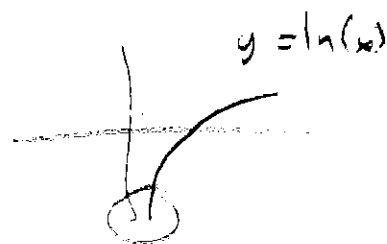
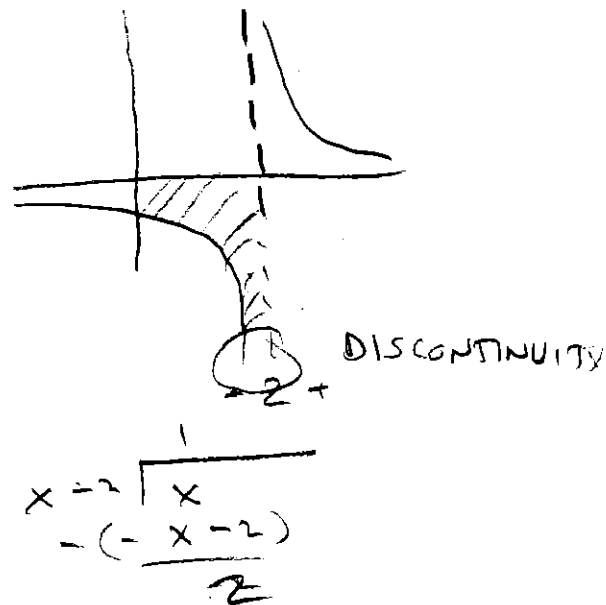
CONVERGES



Example:

$$\begin{aligned}\int_0^2 \frac{x}{x-2} dx &= \lim_{t \rightarrow 2^-} \left[\int_0^t \frac{x}{x-2} dx \right] \\ &= \lim_{t \rightarrow 2^-} \left[\int_0^t 1 + \frac{2}{x-2} dx \right] \\ &= \lim_{t \rightarrow 2^-} \left[x + 2 \ln|x-2| \Big|_0^t \right] \\ &= \lim_{t \rightarrow 2^-} \left[\underbrace{(t^2 + 2 \ln|t-2|)}_0 - (0 + 2 \ln(2)) \right] \\ &\quad \quad \quad \underbrace{\hspace{10em}}_{-\infty}\end{aligned}$$

DIVERGES



If $f(x)$ has a discontinuity at $x = c$ which is **between** a and b , then

$$\int_a^b f(x) dx = \lim_{r \rightarrow c^-} \int_a^r f(x) dx + \lim_{t \rightarrow c^+} \int_t^b f(x) dx$$

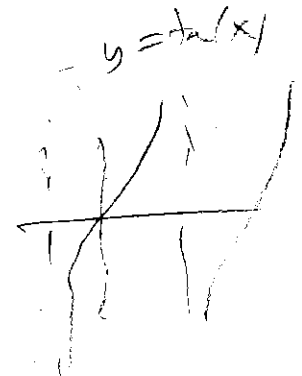
In this case, we say it *converges* only if both limits separately exist and are finite.

Example:

$$\int_0^{\pi} \frac{1}{\cos^2(x)} dx = \int_0^{\pi} \sec^2(x) dx$$

$\cos(x) = 0$ AT $x = \frac{\pi}{2}$

$$\begin{aligned} &= \lim_{r \rightarrow \frac{\pi}{2}^-} \left[\int_0^r \sec^2(x) dx \right] + \lim_{t \rightarrow \frac{\pi}{2}^+} \left[\int_t^{\pi} \sec^2(x) dx \right] \\ &= \lim_{r \rightarrow \frac{\pi}{2}^-} \left[\tan(x) \Big|_0^r \right] + \lim_{t \rightarrow \frac{\pi}{2}^+} \left[\tan(x) \Big|_t^{\pi} \right] \\ &= \lim_{r \rightarrow \frac{\pi}{2}^-} \underbrace{\left[\tan(r) - 0 \right]}_{+\infty} + \lim_{t \rightarrow \frac{\pi}{2}^+} \underbrace{\left[0 - \tan(t) \right]}_{+\infty} \end{aligned}$$



Limits Refresher

1. If stuck, plug in values "near" t .
2. Know your basic functions/values:

$$\lim_{t \rightarrow \infty} \frac{1}{t^a} = 0, \quad \text{if } a > 0.$$

$$\lim_{t \rightarrow \infty} \frac{1}{e^{at}} = 0, \quad \text{if } a > 0.$$

$$\lim_{t \rightarrow \infty} t^a = \infty, \quad \text{if } a > 0.$$

$$\lim_{t \rightarrow \infty} \ln(t) = \infty.$$

$$\lim_{t \rightarrow 0^+} \ln(t) = -\infty.$$

3. For indeterminate forms, use algebra and/or L'Hopital's rule

Examples:

$$\lim_{t \rightarrow 1} \frac{t^2 + 2t - 3}{t - 1} = \lim_{t \rightarrow 1} \frac{(t-1)(t+3)}{(t-1)} = 4$$

$$\lim_{t \rightarrow \infty} \frac{\ln(t)}{t} = \lim_{t \rightarrow \infty} \frac{1/t}{1} = 0$$

$$\lim_{t \rightarrow \infty} t^2 e^{-3t} = \lim_{t \rightarrow \infty} \frac{t^2}{e^{3t}} = \lim_{t \rightarrow \infty} \frac{2t}{3e^{3t}} = \lim_{t \rightarrow \infty} \frac{2}{e^{3t}} = 0$$

Aside:

A few general notes on **comparison**:

Suppose you have two functions $f(x)$ and $g(x)$ such that $0 \leq g(x) \leq f(x)$ for all values.

(a) If $\int_1^{\infty} f(x) dx$ converges,
then $\int_1^{\infty} g(x) dx$ converges.

(b) If $\int_1^{\infty} g(x) dx$ diverges,
then $\int_1^{\infty} f(x) dx$ diverges.

You can verify that

$$\int_1^{\infty} \frac{1}{x^p} dx, \quad \text{converges for } p > 1.$$

$$\int_1^{\infty} e^{px} dx, \quad \text{converges for } p < 0.$$

And you can compare off of these to sometimes quickly tell is something is converging or diverging (without calculating anything)

Ex)

$$\int_1^{\infty} \frac{(\sin(x) + 1)}{x^2} dx$$

$$0 \leq \frac{\sin(x) + 1}{x^2} \leq \frac{2}{x^2}$$

for all $x \geq 1$

$$\text{So } \int_1^{\infty} \frac{\sin(x) + 1}{x^2} dx \leq \int_1^{\infty} \frac{2}{x^2} dx$$



And
THIS
CONVERGES!

SO THIS ALSO CONVERGES